PERGAMON

International Journal of Solids and Structures 38 (2001) 7681–7690

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Rupture energy evaluation for brittle materials

Erast Gaziev *

Institute of Engineering, National Autonomous University of Mexico, Ciudad Universitaria, A.P. 40-472, Coyoacan, 04510 Mexico, D.F., Mexico

Received 17 September 1999

Abstract

The research carried out indicates that there exists a close relationship between rupture energy and the intensity of the stress applied at the moment of failure. This relationship is valid for both the uniaxial and the triaxial strength tests therefore confirming that during the failure process there is a simultaneous mobilization of the tensile and the shear strengths whereas the rupture energy is determined from the distortion energy. © 2001 Published by Elsevier Science Ltd.

Keywords: Distortion energy; Strength criterion; Intensity of stresses and deformations

1. Introduction

The failure process of a material when subjected to loading is always linked to the consumption of energy. Work performed by external forces leads to the accumulation of energy by the constituting elements of the material structure; failure of the material suddenly releases part of the cumulative energy. Processes related to micro and macrocracking of the material dissipate this mechanical energy in terms of heat or acoustic energy. All of these processes occur simultaneously.

Therefore, energy spent by the rupture mechanism represents a parameter that determines the strength and deformability characteristics of a material.

Back in 1855, Beltrami proposed to assume as a shear strength criterion of the material the amount of energy necessary for its deformation; the author however was not successful in this approach. It is a well-known fact that during triaxial hydrostatic compression of a specimen, the material is capable of accumulating a huge amount of energy without evidencing visible indications of failure. Therefore, not all energy spent for deformation becomes determinant but only the component necessary for distorting the specimen. This idea was advanced by J.C. Maxwell in his letter to W. Thomson of 1856: "I have strong reasons for believing that when (the strain energy of distortion) reaches a certain limit then the element will

* Tel.: +52-5622-3426; fax: +52-5616-1514.
E-mail address: egv@pumas.iingen.unam.mx (E. Gaziev).

0020-7683/01/$ - see front matter
PII: S0020-7683(01)00037-3

begin to give way''. We see that Maxwell already had the theory of yielding which we now call *the maximum distortion energy theory*. But he never came back again to this question and his ideas became known only after publication of Maxwell's letters (Timoshenko, 1953). It took engineers a considerable time before they finally returned to this maximum distortion energy theory.

A similar hypothesis was developed by M.T. Huber in 1904 and also independently by R. von Mises en 1913 (Timoshenko, 1953) and as a result it was known as the theory of Huber and Mises. This theory, that was proposed to describe the beginning of a plastic-type behavior of soft steel alloys is based on the fact that the limiting state at a point within a mass is reached when the so-called specific distortion energy becomes equal to a predetermined value. Several trials were made subsequently to develop a theory based on the distortion energy. One of such criteria describing the strength of a polycrystalline material subjected to a multi-axial state of stresses based on the same theory with a satisfactory accuracy of the results obtained was proposed by the author of this paper (Gaziev et al., 1984; Gaziev, 1996; Gaziev and Levtchouk, 1999).

2. Rupture energy evaluation

For analyzing the work performed by the external forces it is convenient to operate with the so-called *intensity of stresses* (Bezukhov, 1961) that is determined from the following equation:

$$\tau_{\mathrm{i}} = \frac{3}{\sqrt{2}}\tau_{\mathrm{oct}} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2} \tag{1}$$

as well as from the *intensity of deformations* that is in turn obtained from the expression:

$$\varepsilon_{\mathrm{i}} = \frac{1}{\sqrt{2}}\sqrt{(\varepsilon_1-\varepsilon_2)^2+(\varepsilon_2-\varepsilon_3)^2+(\varepsilon_1-\varepsilon_3)^2} \tag{2}$$

These two parameters are directly proportional to the square root of the second invariant of the deviatoric stress and of the strain tensors.

When failure occurs, $\tau_{\mathrm{i}} = \tau_{\mathrm{cr}}$. In the case of the uniaxial test τ_{cr} becomes the unconfined compressive strength, $\tau_{\mathrm{cr}} = R_{\mathrm{c}}$, whereas for the ''conventional'' triaxial test (when $\sigma_2 = \sigma_3$) it assumes the value of the peak shear strength, $\tau_{\mathrm{cr}} = (\sigma_1 - \sigma_3)$, just at the time when failure starts.

Work of the external distortion forces when failure occurs or the rupture energy for a unit volume of the specimen can be calculated from,

$$E_{\mathrm{cr}} = \int_0^{\varepsilon_{\mathrm{cr}}} \tau_{\mathrm{i}}(\varepsilon_{\mathrm{i}})\,\mathrm{d}\varepsilon_{\mathrm{i}} \tag{3}$$

For the case of a uniaxial test the moment of rupture is determined from the peak stress obtained. As an example, for the situation presented in Fig. 1, where the stress intensity $\tau_{\mathrm{i}} = \sigma_1$, $\tau_{\mathrm{cr}} = R_{\mathrm{c}} = 51.7$ MPa, and $\varepsilon_{\mathrm{cr}} = 0.0087$; the rupture energy can be calculated as:

$$E_{\mathrm{cr}} = \int_0^{\varepsilon_{\mathrm{cr}}} \sigma_1(\varepsilon_{\mathrm{i}})\,\mathrm{d}\varepsilon_{\mathrm{i}} = 390\ \mathrm{kJ/m^3}$$

For a triaxial test, the moment when failure starts is determined with the phenomenological criterion that considers the first invariant of stresses and the second invariant of deviatoric stress (Gaziev, 1996; Gaziev and Levtchouk, 1999):

$$\frac{\sigma_* + m}{1+m} = \left(\frac{\tau_* - m}{1-m}\right)^n \tag{4}$$

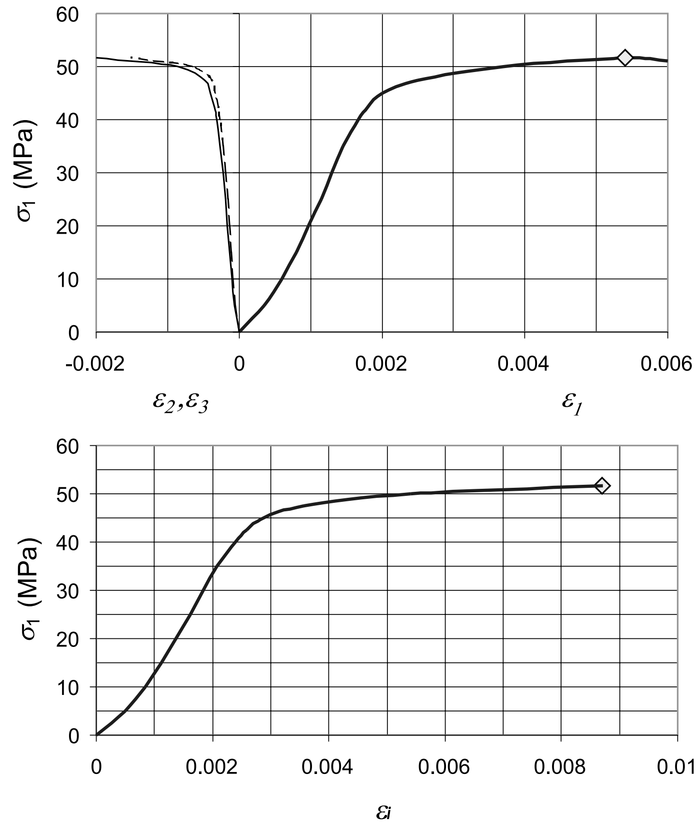

Fig. 1. Stress–strain curves for concrete specimen "c21".

All parameters involved are dimensionless, i.e.:

$$\sigma_* = \frac{\sigma_1 + \sigma_2 + \sigma_3}{R_c} \tag{5}$$

$$\tau_* = \frac{\tau_{cr}}{R_c} = \frac{1}{R_c\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2} \tag{6}$$

$$m = R_t/R_c \tag{7}$$

The expression for "n" in the first approach was derived from experimental studies:

$$n = 1.3 + 0.3\left(\frac{\sigma_2 - \sigma_3}{\sigma_2 + \sigma_3}\right) \tag{8}$$

In all cases when there is no reliable tensile strength data, the suggested criterion allows the evaluation of m, as a parameter of criterion equation (Eq. (4)), by processing experimental data on triaxial state of stresses. Any simple mathematical method can be used.

The results of thorough investigations of Takahashi and Koide (1989), kindly placed at our disposal by Dr. Takahashi (studies of Shirahama and Izumi sandstones, Westerly granite, Yuubari shale and Yamaguchi marble), as well as the experimental results of Bieniawski (1971), Parate (1969) and Vouille and

Laurent (1969) were used by the author for verification of the criterion for different magnitudes of all three principal stresses.

To justify the proposed criterion for different combinations of principal stresses at failure, the linear dependence between the left and right parts of the Eq. (4) were used:

$$X = \frac{\sigma_* + m}{1 + m}$$

$$Y = \left(\frac{\tau_* - m}{1 - m}\right)^n$$

These experimental results are presented in Fig. 2.

Good agreement of the criterion with experimental data is observed over a fairly wide spectrum of the principal stresses.

The peak stress τ_i at the beginning of rupture can be determined from the criterion proposed, i.e.

$$\tau_{i*} = R_t + (R_c - R_t)\left(\frac{\sigma_1 + \sigma_2 + \sigma_3 + R_t}{R_c + R_t}\right)^{1/n} \quad (9)$$

To determine the moment when failure starts, the acting stress τ_i can be divided into its predetermined limiting value.

At the moment when $\tau_i/\tau_{i*} = 1$ the surface of resistance is reached and if the value of τ_i continues to rise the representative point "slides" during a certain time along this surface (the failure process in a three-dimensional state).

Table 1 and Fig. 3 depict the data of a triaxial test performed on a concrete specimen (c3). With the criterion proposed it is possible to determine the time of rupture that occurred when the principal stresses

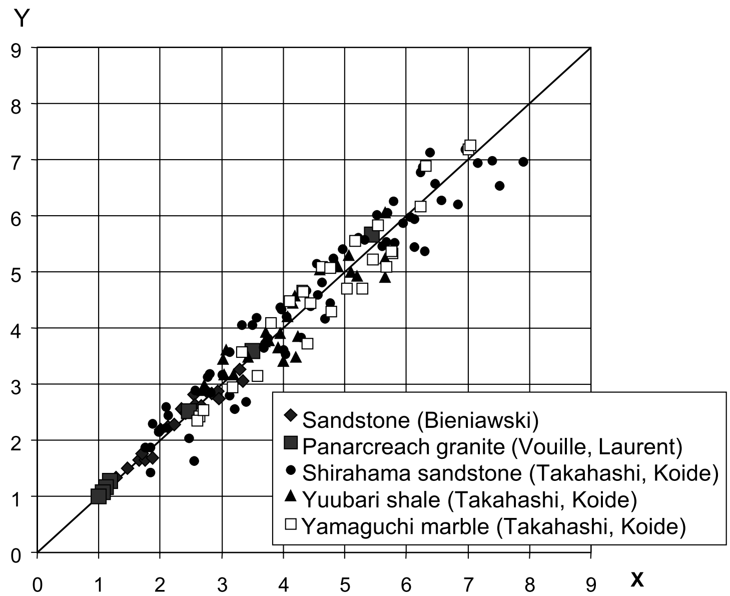

Fig. 2. Comparison of proposed criterion with experimental triaxial tests.

Table 1
Data of the triaxial test of concrete specimen (c3)

σ_1 (MPa)	$\sigma_2 = \sigma_3$ (MPa)	ε_1	$\varepsilon_2 = \varepsilon_3$	ε_i	τ_i (MPa)	τ_{i*} (MPa)	τ_i/τ_{i*}
0	0	0	0	0	0	7.139	0
0.98	0.006	0.00011	-1.5×10^{-5}	0.000125	0.974	8.348	0.117
1.96	0.012	0.00022	−0.00003	0.00025	1.949	9.482	0.205
2.94	0.018	0.00033	-4.5×10^{-5}	0.000375	2.923	10.562	0.277
3.92	0.024	0.00044	−0.00006	0.0005	3.897	11.599	0.336
4.90	0.030	0.00055	-7.5×10^{-5}	0.000625	4.872	12.601	0.387
5.88	0.036	0.00066	−0.00009	0.00075	5.846	13.573	0.431
6.86	0.043	0.00077	−0.00011	0.000875	6.820	14.520	0.470
7.84	0.049	0.00088	−0.00012	0.001	7.795	15.445	0.505
8.82	0.055	0.00099	−0.00014	0.001125	8.769	16.349	0.536
9.80	0.061	0.0011	−0.00015	0.00125	9.743	17.236	0.565
14.71	0.092	0.00165	−0.00023	0.001875	14.614	21.459	0.681
19.61	0.123	0.0022	−0.0003	0.0025	19.485	25.409	0.767
22.55	0.144	0.00253	−0.00035	0.002875	22.405	27.687	0.809
23.53	0.147	0.002642	−0.00036	0.003002	23.382	28.426	0.823
24.51	0.157	0.002757	−0.00038	0.003133	24.353	29.168	0.835
25.49	0.167	0.002876	−0.00039	0.003269	25.324	29.905	0.847
26.47	0.186	0.002998	−0.00041	0.00341	26.284	30.650	0.858
27.45	0.196	0.003126	−0.00044	0.003561	27.255	31.374	0.869
28.43	0.206	0.003259	−0.00046	0.003722	28.225	32.093	0.879
29.41	0.225	0.003399	−0.0005	0.003897	29.186	32.821	0.889
30.39	0.235	0.003547	−0.00054	0.004088	30.157	33.530	0.899
31.37	0.255	0.003705	−0.00059	0.004299	31.118	34.248	0.909
32.35	0.275	0.003873	−0.00067	0.004539	32.078	34.961	0.917
33.33	0.294	0.004053	−0.00073	0.004785	33.039	35.669	0.926
34.31	0.314	0.004246	−0.00082	0.005066	34.000	36.373	0.935
35.29	0.343	0.004454	−0.00093	0.005379	34.951	37.085	0.942
36.27	0.373	0.00468	−0.00105	0.005725	35.902	37.794	0.950
37.25	0.392	0.004923	−0.00119	0.006108	36.863	38.484	0.958
38.24	0.431	0.005183	−0.00134	0.006527	37.804	39.197	0.964
39.22	0.461	0.005469	−0.00153	0.006999	38.755	39.893	0.971
40.20	0.500	0.005785	−0.00174	0.007524	39.696	40.598	0.978
41.18	0.559	0.006133	−0.00197	0.008102	40.618	41.325	0.983
42.16	0.608	0.006525	−0.00222	0.008747	41.549	42.035	0.988
43.14	0.686	0.00696	−0.0025	0.009457	42.451	42.780	0.992
44.12	0.775	0.00746	−0.00279	0.010253	43.343	43.533	0.996
45.10	0.882	0.00804	−0.00311	0.011151	44.216	44.307	0.998
46.08	***0.980***	***0.00871***	***−0.00345***	***0.012161***	***45.098***	***45.065***	***1.000***
46.64	1.049	0.0094	−0.00374	0.01314	45.588	45.511	1.002

reached the values of $\sigma_{1cr} = 46.08$ MPa, $\sigma_{2cr} = \sigma_{3cr} = 0.98$ MPa. The corresponding strains became equal to $\varepsilon_{1cr} = 0.00871$ and $\varepsilon_{2cr} = \varepsilon_{3cr} = -0.00345$. The magnitude of the critical stress was equal to $\tau_{cr} = 45.1$ MPa and that of the critical strain became $\varepsilon_{cr} = 0.012161$.

The corresponding rupture energy resulted equal to

$$E = \int_0^{\varepsilon_{cr}} \tau_i(\varepsilon_i)\,d\varepsilon_i = 340\ \text{kJ/m}^3$$

Table 2 and Fig. 4 present data of other triaxial test results obtained from another concrete specimen (c4). The criterion proposed made it possible to determine the moment of the failure that occurred when the

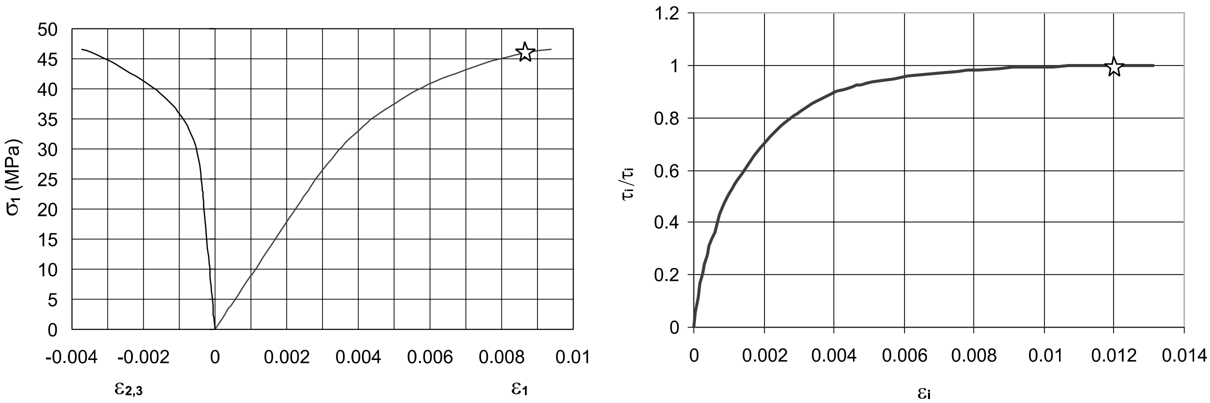

Fig. 3. Stress–strain diagrams for triaxial test of concrete specimen "c3".

principal stresses reached the values of $\sigma_{1cr} = 63.44$ MPa and $\sigma_{3cr} = 2.92$ MPa. The corresponding strain was equal to $\varepsilon_{1cr} = 0.00772$. The magnitude of the critical stress became equal to $\tau_{cr} = 60.47$ MPa and the rupture energy $E_{cr} = 415$ kJ/m^3 (see Tables 2 and 4).

The results of several uniaxial tests carried out by various authors are presented in Table 3 whereas Table 4 contains the results of some triaxial tests performed at the Institute of Engineering of the National Autonomous University of Mexico.

The diagram that relates the rupture energy E_{cr} with the stress intensity τ_{cr} (or the unconfined compression strength in the case of uniaxial tests, R_c) is depicted in Fig. 5. A very reasonable relationship can be observed.

Table 2
Data of the triaxial test of concrete specimen (c4)

σ_1 (MPa)	σ_2 (MPa)	σ_3 (MPa)	ε_1	ε_2	ε_3	ε_i	τ_i (MPa)	τ_{i*} (MPa)	τ_i/τ_{i*}
0	2.47	2.48	0	0	0	0	2.48	13.70	0.181
18.7	2.47	2.48	0.001	−0.00042	−0.0001	0.00129	16.23	29.78	0.545
37.4	2.47	2.48	0.002	−0.0004	−0.0002	0.00231	34.93	43.21	0.808
48.4	2.47	2.48	0.003	−0.0005	−0.0003	0.0034	45.93	50.49	0.910
54.83	2.47	2.48	0.00413	−0.00064	−0.00042	0.00466	52.35	54.59	0.959
55.29	2.48	2.49	0.00426	−0.00068	−0.00048	0.00484	52.8	54.89	0.962
55.76	2.5	2.5	0.00434	−0.00072	−0.00056	0.00498	53.26	55.20	0.965
56.22	2.51	2.51	0.00446	−0.00074	−0.0006	0.00513	53.71	55.50	0.968
56.67	2.53	2.52	0.00457	−0.00074	−0.00064	0.00526	54.15	55.80	0.970
57.12	2.54	2.54	0.00469	−0.00076	−0.00071	0.00543	54.58	56.10	0.973
57.39	2.57	2.55	0.00484	−0.00077	−0.00077	0.00561	54.81	56.30	0.974
57.62	2.59	2.56	0.00498	−0.00079	−0.0008	0.00578	55.03	56.46	0.975
62.46	2.92	2.82	0.00703	−0.00122	−0.00172	0.00851	59.59	59.81	0.996
63.44	***3.03***	***2.92***	***0.00772***	***−0.00133***	***−0.00203***	***0.00942***	***60.47***	***60.54***	***1.000***
64.35	3.16	3.02	0.00846	−0.00149	−0.00234	0.0104	61.26	61.23	1.000
65.33	3.3	3.14	0.00932	−0.00164	−0.00274	0.01155	62.11	61.98	1.002
66.36	3.46	3.27	0.01029	−0.0018	−0.00313	0.01281	62.99	62.78	1.003
67.33	3.63	3.41	0.01131	−0.00198	−0.00358	0.01416	63.81	63.55	1.004
68.47	3.84	3.57	0.01248	−0.00219	−0.00409	0.01571	64.76	64.45	1.005
68.74	4.04	3.72	0.01364	−0.00242	−0.00452	0.01721	64.86	64.82	1.006

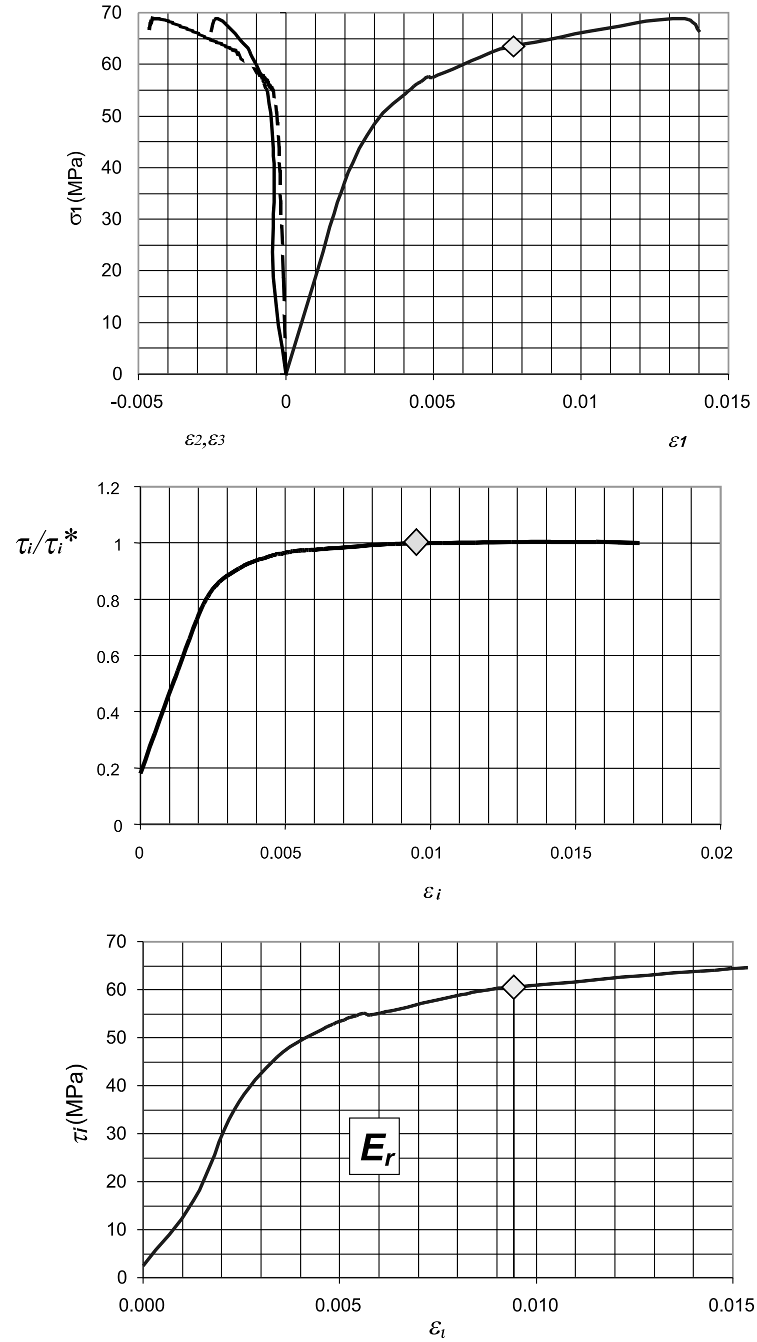

Fig. 4. Stress–strain diagrams for triaxial test of concrete specimen "c4".

The fact that the rupture energy for both the uniaxial and the triaxial tests is described in terms of the same relationship based on the distortion energy evidences that the failure process of rock materials is

Table 3
Data presented on the diagram $E_{cr} = f(\tau_{icr})$ (Fig. 5) – uniaxial tests

Mark on the diagram	Rock type	R_c (MPa)	ε_{icr}	E_{cr} (kJ/m^3)	References
1	Diabase breccias	130–153	0.0143–0.0175	849–1139	Pininska and Lukaszewski (1991)
2	Diabases with quartz	87–128	0.0121–0.0141	474–788	
3	Diabases	60–76	0.0102–0.0123	336–432	
4	Metadiabase breccias	90–95	0.0108–0.0121	416–528	
5	Metadiabases with quartz	53–58	0.0084–0.0105	217–265	
6	Metadiabases	30–35	0.0070–0.0094	157–187	
7	Quartz shists	40–45	0.0038–0.0089	191–282	
8	Chlorite shists	17–24	0.0071–0.0089	89–183	
9	Breccias of shists	18–28	0.0115–0.0133	59–153	
S	Sandstone	143	0.0084	601	(Kawamoto and Saito (1991)
T	Tuff	44	0.0074	165	
DC	Diabase (coggins)	341	0.00738	1531	Miller (1965)
B	Basalt (lower granite)	223	0.00761	1092	
c21	Concrete	51,7	0.0087	390	Author[a]
g	Gypsum	12	0.0029	19	

[a] Experimental studies were realized by author and assisted by Vadim Levtchouk and César D. Valdés Mosqueda.

Table 4
Data presented on the diagram $E_{cr} = f(\tau_{icr})$ (Fig. 5) – triaxial tests[a]

Mark on the diagram	Material	Stresses at failure	τ_{cr} (MPa)	ε_{icr}	E_{cr} (kJ/m^3)
c2	Concrete $R_c = 45$ MPa $R_t = 2.9$ MPa	$\sigma_{1cr} = 63.3$ MPa $\sigma_{2cr} = \sigma_{3cr} = 2.9$ MPa	43.35	0.00612	180
c3	Cement mortar $R_c = 39.4$ MPa $R_t = 2.7$ MPa	$\sigma_{1cr} = 46.1$ MPa $\sigma_{2cr} = \sigma_{3cr} = 1.0$ MPa	45.1	0.01212	340
c4	Concrete $R_c = 45$ MPa $R_t = 3.1$ MPa	$\sigma_{1cr} = 63.44$ MPa $\sigma_{2cr} = \sigma_{3cr} = 2.97$ MPa	60.47	0.00942	415
c5	Cement mortar $R_c = 39.4$ MPa $R_t = 2.7$ MPa	$\sigma_{1cr} = 61$ MPa $\sigma_{2cr} = 4.65$ MPa $\sigma_{3cr} = 3.13$ MPa	57.14	0.01230	381

[a] Experimental studies were realized by author and assisted by Vadim Levtchouk and César D. Valdés Mosqueda.

induced by the joint action of normal and shear stresses. The normal tensile stresses develop the conditions necessary for the failure to occur (at a macroscopic level) under the action of shear stresses. The experimental work on the failure mechanism during shear performed in specimens, in models and at the field showed that this process starts with the development of tension-induced microcracks at the zone where the shear stresses will occur (Miller, 1965; Vouille and Laurent, 1969; Fishman and Gaziev, 1974; Gaziev et al., 1975; Pininska and Lukaszewski, 1991; Kawamoto and Saito, 1991). The same conclusion was derived by Martin and Chandler (1994) according to which tensile and shear strengths develop simultaneously.

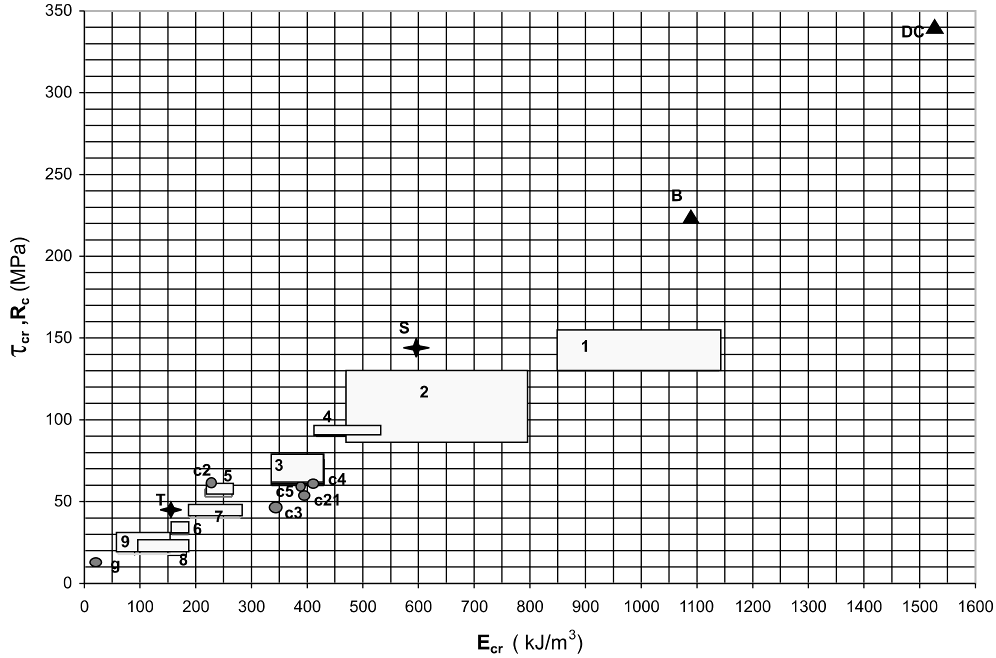

Fig. 5. Diagram of strength–energy of rupture relation for different rocks (see Tables 3 and 4).

3. Conclusions

1. The distortion energy spent by the rupture mechanism represents a parameter that determines the strength and deformability characteristics of a material. There exists a close relationship between the rupture energy and the intensity of the stress applied at the moment of failure (see Fig. 5). This relationship is valid for both the uniaxial and the triaxial tests therefore confirming that the tensile and the shear strengths are simultaneously mobilized and the rupture energy is determined from the distortion energy.
2. It has been proposed to use for analyzing the rupture energy of brittle polycrystalline materials the stress and deformation intensities expressed by the equations Eqs. (1) and (2).
3. In the case of a uniaxial test the moment of failure is determined from the peak stress obtained (the intensity of stress $\tau_i = \sigma_1, \tau_{cr} = R_c$); on the other hand, for a triaxial test the moment of failure is calculated with the criterion proposed (Eq. (4)), namely:

$$\frac{\sigma_* + m}{1 + m} = \left(\frac{\tau_* - m}{1 - m}\right)^n$$

This strength criterion permits the determination of the strength for polycrystalline materials (rock, concrete, etc.) in any state of stress (Gaziev, 1996; Gaziev and Levtchouk, 1999).

Acknowledgements

The author gratefully acknowledges the support for this work received from the researcher of the Institute of Engineering of the National Autonomous University of Mexico, Prof. Jesús Alberro Aramburu, and the practical assistance in the experimental studies received from the collaborators of the same Institute Vadim Levtchouk and César David Valdés Mosqueda.

References

Bezukhov, N.I., 1961. The bases of the theory of elasticity, plasticity and creep (in Russian). Editorial "Vysshaya Shkola" (High School), Moscow.

Bieniawski, Z.T., 1971. Deformational behaviour of fractured rock under multiaxial compression. In: Te'eni M. (Ed.), Structure, Solid Mechanics and Engineering Design, Part 1. Wiley-Interscience, London, pp. 589–598.

Fishman, Yu.A., Gaziev, E.G., 1974. In situ and model studies of rock foundation failure in concrete block shear tests. 3rd International Congress of the ISRM, vol. II-B. Denver, USA, pp. 879–883.

Gaziev, E.G., Fishman, Yu.A., Erlikhman, S.A., 1975. Shear strength. Studies on models (in Russian). First Baltic Conference on Soil Mechanics and Foundation Engineering, vol. 3. Gdansk, Poland, pp. 397–405.

Gaziev, E., Morozov, A., Chaganian, V., 1984. Comportement expérimental des roches sous contraintes et déformations triaxiales. Revue Française de Géotechnique Paris (29), pp. 43–48.

Gaziev, E., 1996. Criterio de resistencia para rocas y materiales frágiles policristalinos. Segunda Conferencia Magistral "Raúl J. Marsal", Sociedad Mexicana de Mecánica de Rocas, Mexico.

Gaziev, E., Levtchouk, V., 1999. Strength characterization for rock under multiaxial stress states. 9th International Congress on Rock Mechanics, Paris, France.

Kawamoto, T., Saito, T., 1991. The behavior of rock-like materials in some controlled strain states. Seventh International Congress on Rock Mechanics, vol. 1., Aachen, Germany, pp. 161–166.

Martin, C.D., Chandler, N.A., 1994. The progressive fracture of Lac du Bonnet granite. Int. J. Rock Mech. Min. Sci. Geomech. Abstr. 31 (6), 643–659.

Miller, R.P., 1965. Engineering classification and index properties for intact rock. Thesis doctoral, University of Illinois, Urbana.

Parate, N.S., 1969. Critère de rupture des roches fragiles. Annales de l'Institut Technique du Batiment et des Travaux Publiques Paris (253), pp. 149–160.

Pininska, J., Lukaszewski, P., 1991. The relationships between post-failure state and compression strength of Sudetic fractured rocks. Bull. Int. Assoc. Eng. Geol.43, 81–86.

Takahashi, M., Koide, H., 1989. Effect of the intermediate principal stress on strength and deformation behavior of sedimentary rocks at the depth shallower than 2000 m. Proceedings of the ISRM-SPE International Symposium "Rock at Great Depth", Pau, France, pp. 19–26.

Timoshenko, S.P., 1953. History of Strength of Materials. McGraw-Hill, New York, pp. 368–371.

Vouille, G., Laurent, D., 1969. Etude de la courbe intrinsèque de quelques granites. Revue de l'Industrie Minérale, Paris, Numero spécial, 15 juillet 1969, pp. 25–28.